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QUASI-STEADY STATE TEMPERATURE DISTRIBUTION IN FINITE REGIONS WITH PERIODICALLY-VARYING BOUNDARY CONDITIONS

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Abstract—Reed and Mullineux applied in [3] a semi-numerical procedure for determining the quasi-steady state solution of periodically varying phenomena. In the present study this problem is reduced to a system of matrix equations without using any approximation. The numerical results are compared with those of [3].

NOMENCLATURE

$A_m(N)$, $B_m(N)$, boundary coefficient functions defined on S;
C_m, $p \times 1$ matrices of which c_{mj} are elements derived in the text;
D_m, $p \times p$ matrices of which d_{mji} are elements derived in the text;
$f_m(M)$, quasi-steady state distribution function in V;
i, $1, 2, 3, \ldots, \infty$;
$k_m(M)$, $w_m(M)$, $\rho_m(M)$, prescribed functions defined in V;
M, point in V;
m, 1 or 2;
N, point on S;
n, outward normal of S;
$P_m(M, \tau)$, internal source functions per unit time per unit volume of V;
p, number of transient terms in infinite series solutions;
S, boundary of V;
$T_m(M, \tau)$, unsteady temperature distribution;
V, finite region of arbitrary geometry;
$\beta_m(\tau)$, $\varphi_m(\tau)$, prescribed functions defined in τ;
τ, time variable;
$\Omega_m(N, \tau)$, source functions on S;
$\psi_{mi}(M)$, eigenfunctions in M-space;
μ_{mi}, eigenvalues;
div( ), divergence operation in M-space;
grad( ), gradient vector in M-space.

Dimensionless criteria

$Bi = \alpha l$, Biot number;
$Fo = \dfrac{\tau}{kl^2}$, Fourier number;
$z = x/l$, dimensionless coordinate;
$T_m(z, Fo) = \theta/\theta_0$, dimensionless temperature;

where

θ, temperature difference;
θ_0, fixed temperature difference at one end of cylinder;
x, distance along cylinder from this end;
l, length of cylinder;
m, contact ($m = 1$) and separation ($m = 2$) times;
k, α, thermal constants of cylinder and film, respectively.

1. INTRODUCTION

USING an analogue computer Howard and Sutton [1, 2] investigated the heat transfer through two bars, the surfaces of which are meeting and separating according to a regular cycle.

Recently Reed and Mullineux [3] are discussing again this problem. They solve the equation of heat conductivity separately for the two intervals of one cycle. The analytical solutions, obtained for these two cases are not new [4] and in Appendix of [3] are repeated well-known facts from Ozisik's textbook [5].

The essence of the proposed by Reed and Mullineux [3] semi-numerical procedure is to divide the length of the bar into s equal intervals and using the trapezoidal approximation rule to obtain a system of two matrix equations for the determination of the two temperature distributions in the moments of closure and separation in the points of approximation.

The authors of [3] mark that "the problem is simple to define but not simple to solve". As a matter of fact as will be shown in the present study, the problem is easily reduced to a system of matrix equations without using any approximation.

The mathematical problem, defined in [3], is analogous to the one, defined in the theory of regenerators. As early as 1928 Heiligenstädt wrote down a set of equations for the cyclic steady state of a regenerator and indicated an approximation solution [6]. Consequently, the general method, presented here, can be useful in studying some of the problems defined in [7].

2. STATEMENT AND SOLUTION OF THE PROBLEM

Let us suppose that in a finite region of arbitrary geometry take place subsequently two independent processes of transfer, described through the differential equations

$$\varphi_m(\tau)w_m(M)\frac{\partial T_m(M,\tau)}{\partial\tau} = \operatorname{div}[k_m(M)\operatorname{grad}T_m(M,\tau)]+[\beta_m(\tau)w_m(M)-\rho_m(M)]\times T_m(M,\tau)+P_m(M,\tau),$$
$$0\leqslant\tau\leqslant\tau_m,\qquad m=1 \text{ or } 2 \quad (1)$$

subject to the following conditions:

$$T_m(M,0)=f_m(M) \quad (2)$$

$$A_m(N)\frac{\partial T_m(N,\tau)}{\partial n}+B_m(N)T_m(N,\tau)=\Omega_m(N,\tau). \quad (3)$$

Since the quasi-steady state is reached, the final temperature distribution for an arbitrary interval turns out to be the initial distribution for the next one. Consequently, one can write down

$$T_m(M,\tau_m)=f_{3-m}(M). \quad (4)$$

It is necessary to find out the potentials $T_m(M,\tau)$ and the unknown distributions $f_m(M)$.

The solution of equation (1) under the conditions (2) and (3) is obtained in [8] and has the form:

$$T_m(M,\tau)=\sum_{i=1}^{\infty}G_{mi}\psi_{mi}(M)\times e^{-h_{mi}(\tau)}\left\{\int_V w_m(M)\psi_{mi}(M)f_m(M)\,dV+\int_0^{\tau}\frac{g_m(\tau)}{\varphi_m(\tau)}e^{h_{mi}(\tau)}\,d\tau\right\} \quad (5)$$

where

$$G_{mi}=\left\{\int_V w_m(M)\psi_{mi}^2(M)\,dV\right\}^{-1} \quad (6)$$

$$h_{mi}(\tau)=\int_0^{\tau}\frac{\mu_{mi}^2-\beta_m(\tau)}{\varphi_m(\tau)}\,d\tau \quad (7)$$

$$g_m(\tau)=\int_S k_m(N)\Omega_m(N,\tau)\frac{\psi_{mi}(N)-\dfrac{\partial\psi_{mi}(N)}{\partial n}}{A_m(N)+B_m(N)}\,dS+\int_V\psi_{mi}(M)P_m(M,\tau)\,dV. \quad (8)$$

In expressions (5)–(8) μ_{mi} and ψ_{mi} are the eigenvalues and eigenfunctions of Sturm–Liouville's problem, which are supposed to be known.

Having in mind (4), from the solution (5) it follows that:

$$f_{3-m}(M)=\sum_{i=1}^{\infty}G_{mi}\psi_{mi}(M)\times e^{-h_{mi}(\tau_m)}\left\{\int_V w_m(M)\psi_{mi}(M)f_m(M)\,dV+\int_0^{\tau_m}\frac{g_m(\tau)}{\varphi_m(\tau)}e^{h_{mi}(\tau)}\,d\tau\right\}. \quad (9)$$

To determine the potentials $T_m(M,\tau)$, it is sufficient to know only the expressions

$$x_{mi}=\int_V w_m(M)\psi_{mi}(M)f_m(M)\,dV. \quad (10)$$

In order to find them, one can multiply (9) by $w_{3-m}(M)\psi_{3-mj}(M)$ and integrate, after which one obtains

$$x_{3-m,j}=c_{mj}+\sum_{i=1}^{p}d_{mji}x_{mi},\qquad (p\to\infty) \quad (11)$$

where

$$c_{mj}=\sum_{i=1}^{p}d_{mji}\int_0^{\tau_m}\frac{g_m(\tau)}{\varphi_m(\tau)}e^{h_{mi}(\tau)}\,d\tau \quad (12)$$

$$d_{mji}=G_{mi}\int_V w_{3-m}(M)\psi_{3-m,j}(M)\times\psi_{mi}(M)\,dV\,e^{-h_{mi}(\tau_m)}. \quad (13)$$

The system (11) can be written in a matrix form

$$X_{3-m}=C_m+D_mX_m \quad (14)$$

and has the following solution [3]:

$$X_m=(I-D_{3-m}D_m)^{-1}(D_{3-m}C_m+C_{3-m}) \quad (15)$$

where I is the unit matrix.

Consequently, from (15) one can calculate the expressions (10) and after that from (5) one can obtain the unknown distributions $T_m(M,\tau)$ and $f_{3-m}(M)$.

The same method is easily applied to the solutions, given in [8], for both cases: when $\rho(M)=0$ and $B(N)=0$, and when the convergence of the series is improved through the pseudo-steady solution of order m. An analogical approach can be applied to a region composed of q subregions, which periodically come in and out of contact. But in this case it is necessary to solve a system of $q+1$ matrix equations.

3. THE HEAT FLOW PROBLEM OF [3]

As an application of the general theory consider the special case of [3], which can be written in the dimensionless form:

$$\frac{\partial T_m(z,Fo)}{\partial Fo} = \frac{\partial^2 T_m(z,Fo)}{\partial z^2},$$

$$0 \leqslant Fo \leqslant Fo_m, \quad 0 \leqslant z \leqslant 1, \quad (16)$$

$$T_m(z,0) = f_m(z) \quad (17)$$

$$T_m(0,Fo) = 1 \quad (18)$$

$$\frac{\partial T_m(1,Fo)}{\partial z} + (2-m)BiT_m(1,Fo) = 0 \quad (19)$$

$$T_m(z,Fo_m) = f_{3-m}(z). \quad (20)$$

The analytical solution of equation (16) under conditions (17)–(19) is easily obtained as a very special case of the solution given in [8]. We prefer to take it directly from [4]:

$$T_m(z,Fo) = 1 - \frac{(2-m)Bi}{1+(2-m)Bi} z + \sum_{i=1}^{\infty} 2\left\{1 + \frac{(2-m)Bi}{\mu_{mi}^2 + [(2-m)Bi]^2}\right\}^{-1} \times \sin(\mu_{mi} z)\, e^{-\mu_{mi}^2 Fo} \times \left\{\int_0^1 \sin(\mu_{mi} z) f_m(z)\,dz - \frac{1}{\mu_{mi}}\right\} \quad (21)$$

where μ_{mi} are the roots of the characteristic equation

$$\cos\mu_m + (2-m)Bi\frac{\sin(\mu_m)}{\mu_m} = 0. \quad (22)$$

The solutions (21), in contrast to those given in (3), have better convergence of the series.

After introducing the unknowns

$$x_{mi} = \int_0^1 \sin(\mu_{mi} z) f_m(z)\,dz \quad (23)$$

and multiplying equation (21) by $\sin(\mu_{mj} z)$, followed by an integration over the interval from 0 to 1 and using equation (20), one obtains an equation, identical to (11), where

$$c_{mj} = \frac{1}{\mu_{3-m,j}}\left\{1 - \cos\mu_{3-m,j} - \frac{(2-m)Bi}{1+(2-m)Bi} \times \left(\frac{\sin\mu_{3-m,j}}{\mu_{3-m,j}} - \cos\mu_{3-m,j}\right)\right\} - \sum_{i=1}^{p}\frac{1}{\mu_{mi}} d_{mji} \quad (24)$$

$$d_{mji} = 2\left\{1 + \frac{(2-m)Bi}{\mu_{mi}^2 + [(2-m)Bi]^2}\right\}^{-1} \times \frac{\mu_{mi}\cos\mu_{mi}\sin\mu_{3-m,j} - \mu_{3-m,j}\cos\mu_{3-m,j}\sin\mu_{mi}}{\mu_{3-m,j}^2 - \mu_{mi}^2} \times e^{-\mu_{mi}^2 Fo_m}. \quad (25)$$

In (3) there is a numerical example, which corresponds to the case: $Bi = 20$, $Fo_1 = Fo_2 = 0{\cdot}078125$. On the base of equations (21)–(25) this case was calculated again. In the calculations the order p of the matrices C and D was increased consequently up when five symbols after the decimal point in the distributions $f_m(z)$ were repeated. It was found that the results for $p = 3$ and $p = 4$ coincide. They are given in Table 1 and fully correspond to the data, presented in the figures in [3].

Table 1

z	$f_2(z)$	$f_1(z)$
0·00	1·00000	1·00000
0·05	0·96014	0·95829
0·10	0·92022	0·91664
0·15	0·88016	0·87511
0·20	0·83987	0·83378
0·25	0·79924	0·79275
0·30	0·75816	0·75218
0·35	0·71647	0·71223
0·40	0·67401	0·67313
0·45	0·63060	0·63515
0·50	0·58605	0·59859
0·55	0·54019	0·56381
0·60	0·49286	0·53118
0·65	0·44394	0·50110
0·70	0·39335	0·47398
0·75	0·34107	0·45021
0·80	0·28716	0·43018
0·85	0·23175	0·41421
0·90	0·17502	0·40260
0·95	0·11725	0·39554
1·00	0·05875	0·39317

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CHAMP DES TEMPERATURES QUASI-STATIONNAIRE DANS UN DOMAINE FINI AVEC LES CONDITIONS AUX LIMITES PERIODIQUES

Résumé—Reed et Mullineux appliquent en [3] des procédés semi-numériques pour trouver une solution quasi-stationnaire de phénomènes périodiques. Dans cet article le problème est réduit à une système matriciel, que l'on résout sans approximation. Les résultats numériques sont comparés à ceux de [3].

QUASISTATIONÄRE TEMPERATURVERTEILUNG IN EINEM BEGRENZTEN GEBIET FÜR PERIODISCH VERÄNDERLICHE RANDBEDINGUNGEN

Zusammenfassung—Reed und Mullineux haben in [3] ein halbnumerisches Verfahren zur Bestimmung der quasistationären Lösung der periodisch veränderlichen Phänomene verwendet. In der vorliegenden Arbeit wird dieses Problem ohne irgendeine Approximation durch ein System aus Matrizengleichungen dargestellt. Die numerischen Ergebnisse werden mit den in [3] erhaltenen verglichen.

КВАЗИСТАЦИОНАРНОЕ РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ В КОНЕЧНОЙ ОБЛАСТИ ПРИ ПЕРИОДИЧЕСКИХ ВОЗДЕЙСТВИЯХ

Аннотация — В работе [3] Рид и Мюллино использовали численно-аналитический метод для нахождения квазистационарного решения в периодических процессах. В этой статье задача без помощи каких-либо аппроксимаций сводится к системе матричных уравнений. Численные результаты сравниваются с данными работы [3].